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Neutrino trapping in rotating matter

S N Guha Thakurta
Department of Physics, Presidency College, Calcutta-700073, India

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Abstract. The paper considers circular null geodesics and turning points for equitorial orbits in the Hogan metric for rotating matter.

1. Introduction

In recent years, there has been some interest in the study of null geodesics within matter distributions. The motivation is to investigate the possibility of trapping neutrinos within stellar objects and this might conceivably be of some significance in the interpretation of the results of the experiments by Davies *et al* (Kuchowicz, 1976).

Following the discovery of trapped null orbits in the Schwarzschild interior metric and some non-static matter distributions (de Felice, 1969; Kuchowicz, 1974), Collas and Lawrence (1976) (herein after referred to as CL) have considered the trapped null geodesics in an interior metric for rotating matter given by Gürses and Gürsey (1975). Their principal conclusion was that the presence of rotation in the background matter increases or decreases the trapping, according as the neutrino angular momentum is in the opposite or same sense as that of the background matter. This conclusion seems interesting as it would enable, in principle, a study of the rotation inside a source from observations on the neutrinos escaping from it.

However, the solution of Gürses and Gürsey possesses some unphysical features, as pointed out by CL, who showed that at least one of the principal stresses is negative if the energy density is positive and the Hawking–Penrose condition is violated. Further, with CL's choice of a function $f(r)$, left arbitrary by Gürses and Gürsey, the field had a singularity at the origin $r = 0$, with the matter density becoming arbitrarily large there.

To the present author, it seemed worthwhile to investigate this important question of trapping for more acceptable rotating matter metrics. One such metric has very recently been proposed by Hogan (1976). This metric has several desirable features—the energy stress-tensor is consistent with the Hawking–Penrose condition, and in the non-rotating limit, the metric passes over to the Schwarzschild interior solution, while the Gürses and Gürsey solution has no acceptable non-rotating limit.

However, the Hogan metric embedded in the exterior Kerr metric requires a surface mass distribution at the boundary which is an obvious shortcoming of the solution.

Hogan's line element is

$$ds^2 = (\Sigma/x)\, dr^2 + \Sigma\, d\theta^2 + (r^2 + a^2)\sin^2\theta\, d\phi^2 - dt^2 + (1-f)(dt + a\sin^2\theta\, d\phi)^2 \qquad (1)$$

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where

$$\Sigma = r^2 + a^2 \cos^2\theta, \qquad x = r^2 - qR^2\Sigma + a^2,$$
$$f = [\tfrac{3}{2}\sqrt{(1-qb^2)} - \tfrac{1}{2}\sqrt{(1-qR^2)}]^2, \qquad R = r^{-1}\Sigma, \qquad q = 2mb^{-3}, \tag{2}$$

b is a constant and $b > 2m$. The metric matches continuously to the Kerr metric

$$ds^2 = (\Sigma/\Delta)\,dr^2 + \Sigma\,d\theta^2 + (r^2+a^2)\sin^2\theta\,d\phi^2 - dt^2 + (2mr/\Sigma)(dt + a\sin^2\theta\,d\phi)^2 \tag{3}$$

with $\Delta = r^2 - 2mr + a^2$.

On the closed two-surface

$$r = \frac{b}{2}\left\{1 + \sqrt{\left(1 - 4\frac{a^2}{b^2}\cos^2\theta\right)}\right\} \tag{4}$$

it being assumed that

$$(\tfrac{1}{2}b)^2 > m^2 \geq a^2. \tag{4$'$}$$

The equitorial boundary is thus a circle of radius $r = b$.

The line element (1) reduces to the interior Schwarzschild metric for a homogeneous sphere of radius $r = b$, when $a = 0$. It may be noted that for $a \neq 0$, if one approaches the origin $r = 0$, from a direction $\theta \neq \pi/2$, the metric (1) ceases to be real and also has an apparent singularity; however, with $\theta = \pi/2$, at the limit $r \to 0$, the metric remains regular and real.

2. Circular orbits in the equitorial plane

CL have found that for the Gürses and Gürsey metric, the Hamilton–Jacobi equation admits an integration by the method of separation of variables as has been shown previously by Carter (1968) for the Kerr metric. For the Hogan metric, however, the variables θ and r cannot be separated (cf the case of Tomimatsu–Sato metric (1973)) and hence unlike the CL case, the radial equation cannot in general be reduced to quadrature. In the present discussion, we have limited our consideration to geodesics in the equitorial plane ($\theta = \pi/2$). We shall first consider circular orbits which allow a simple analytical discussion.

For a circular orbit in the equitorial plane, we have $\theta = \pi/2$, $\dot\theta = 0$, $r =$ constant, $\dot r = 0$, and hence the geodesic equation gives

$$\frac{1}{2}\frac{x}{\Sigma}[2r - a^2 f']\dot\phi^2 - a\frac{x}{\Sigma}f'\dot\phi\dot t - \frac{1}{2}\frac{x}{\Sigma}f'\dot t^2 = 0 \tag{5}$$

where the dot denotes differentiation with respect to an affine parameter and the prime denotes differentiation with respect to r. We have from equation (5),

$$\frac{d\phi}{dt} = \frac{\dot\phi}{\dot t} = \pm\frac{1}{(2r/f')^{1/2} \mp a}. \tag{6}$$

Again for a null geodesic, $ds = 0$, so that

$$[r^2 + a^2 + (1-f)a^2]\dot\phi^2 + (1-f)2a\dot\phi\dot t - f\dot t^2 = 0. \tag{7}$$

Eliminating $\dot{\phi}$ with the help of equation (6), equation (7) may be put in the form

$$r^2 - f^{1/2}\frac{[1-(2mr^2/b^3)]^{1/2}}{(m/b^3)} = -2a\left(\frac{m}{b^3}\right)^{-1/2}\frac{[1-(2mr^2/b^3)]^{1/4}}{f^{1/4}}\begin{bmatrix}1\\2f-1\end{bmatrix}. \tag{8}$$

Writing

$$2mr^2/b^3 = \cos^2\psi \tag{9}$$

we get

$$(A-\tfrac{1}{2}\sin\psi)^{1/2}(\tfrac{1}{2}-A\sin\psi) = -2a\left(\frac{m}{b^3}\right)^{1/2}(\sin\psi)^{1/2}\begin{bmatrix}1\\2f-1\end{bmatrix} \tag{10}$$

where

$$A \equiv \tfrac{3}{2}[1-(2m/b)]^{1/2} = \tfrac{3}{2}\sin\psi_b \tag{11}$$

so that

$$f = (A-\tfrac{1}{2}\sin\psi)^2. \tag{12}$$

The upper and lower quantities in the boxes in equation (8) and (10) correspond respectively to the upper and lower signs in equation (6). We may note also that ψ decreases from $\pi/2$ at $r=0$ to $\psi_b = \cos^{-1}(2m/b)^{1/2}$ at the boundary.

The following tables, obtained by direct computation from equation (10) show the values of m/b and a/b. It may be noted that the value $m/b = 4/9 \approx 0{\cdot}444$ corresponds to $f=0$ at $r=0$ and thus the limiting concentration for regular solutions.

For such a case (very 'dense' source, $m/b \approx 0{\cdot}442$) the circular orbit in the non-rotating case ($a=0$, i.e. Schwarzschild interior metric) lies well within the source ($r/b \approx 0{\cdot}20$,—not given in the table) while with the decreasing values of m/b ('diffuse' sources), the circular orbits shift outwards and in the case $m/b = 1/3 \approx 0{\cdot}333$ it lies just at the boundary of the source.

We have computed for the dense source ($m/b = 0{\cdot}40$), the circular orbit obtained for the non-rotating case ($a=0$) at $r/b \approx 0{\cdot}72$, showing that in this region, even a small change of m/b causes a fairly large change in r/b.

When $a \neq 0$, the table shows that circular orbits are of decreasing or increasing radius according as the particle angular velocity is in the same or opposite sense to that of the source. Further, the tables show that there exists bounds to the value of a/b (for a given m/b) above which no equitorial circular orbit is possible. The bounds are however different for the case of co- and counter-rotation.

The table lists the circular orbits under the sub-heading co-rotation and counter-rotation. Equation (6) shows $\dot{\phi}$ is of the same sign as 'a' (i.e. there is co-rotation) if we use the upper sign and $(2r/f')^{1/2} > a$. In all other cases there will be counter-rotation (i.e. $\dot{\phi}$ is of the opposite sign to 'a'). The condition for co-rotation thus becomes, on substitution for f',

$$\left[\frac{\sin\psi}{(A-\tfrac{1}{2}\sin\psi)(m/b^3)}\right]^{1/2} > a. \tag{13}$$

However, from equation (10) (with the upper value), the limiting value of 'a' for a circular orbit in the equitorial plane is given by ($r \to 0$, $\psi \to \pi/2$)

$$(A-\tfrac{1}{2})^{3/2} = 2a(m/b^3)^{1/2} \tag{14}$$

which gives with $m/b = 0{\cdot}40, 0{\cdot}30$, $a/b = 0{\cdot}056$, and $0{\cdot}274$ respectively (values given in the table) where the inequality (13) is satisfied, showing that all the cases with the upper value correspond to co-rotation.

Table 1. Showing the circular orbit for a 'dense' source.

Sense	m/b	a/b	r/b
	0·40	0	0·72
		0·05	0·31
Co-rotation		0·055	0·05
		0·056	0
	0·40		
		0·05	0·90
		0·10	0·94
Counter rotation		0·20	0·96
		0·25	0·99
		0·26	1·0

Table 2. Showing circular orbit for a 'diffuse' source.

Sense	m/b	a/b	r/b
		0·10	0·99
		0·20	0·76
Co-rotation			
		0·25	0·47
		0·274	0
	0·30		
Counter rotation		No circular orbit within the boundary.	

3. Turning points in the case of a general equitorial path

For the Hogan metric (1), the radial equation of motion for particles describing null geodesics may be written as (cf Carter; CL)

$$r^2\dot{r} = \mp\sqrt{xR} \tag{15}$$

where R is given by

$$R = \frac{r^2}{r^2 f + a^2}\{[r^2 + a^2 + (1-f)a^2]E^2 + 2a(1-f)E\Phi - f\Phi^2\}. \tag{16}$$

E and Φ are identified as the energy and the Z-component of angular momentum of the test particle, respectively, and are explicitly given by

$$\begin{aligned} E &= -p_t = -f\dot{t} + (1-f)a \sin^2\theta\dot{\phi} \\ \Phi &= [(r^2 + a^2)\sin^2\theta + (1-f)a^2\sin^4\theta]\dot{\phi} + (1-f)a\sin^2\theta\dot{t} \end{aligned} \tag{17}$$

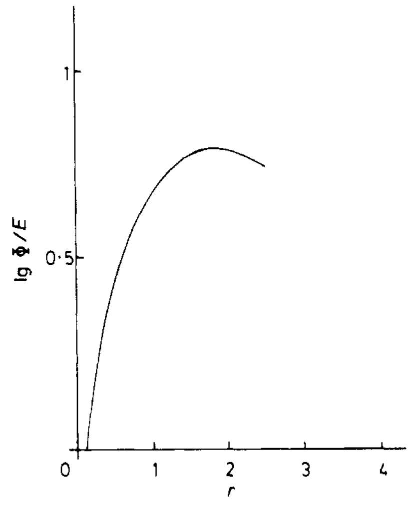

Figure 1. For a non-rotating 'dense' source ($a = 0$, $m/b = 0{\cdot}40$).

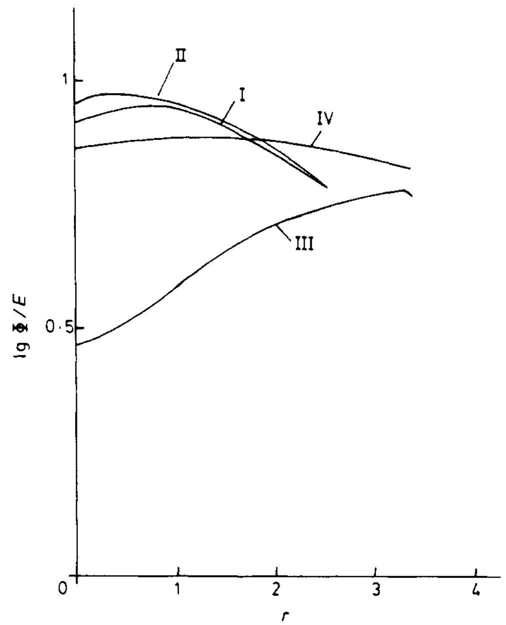

Figure 2. For co-rotation with different values of a/b and m/b. I—$a/b = 0{\cdot}05$ in a 'dense' source ($m/b = 0{\cdot}40$). II—$a/b = 0{\cdot}055$ in a 'dense' source ($m/b = 0{\cdot}40$). III—$a/b = 0{\cdot}10$ in a 'diffuse' source ($m/b = 0{\cdot}30$). IV—$a/b = 0{\cdot}25$ in a 'diffuse' source ($m/b = 0{\cdot}30$).

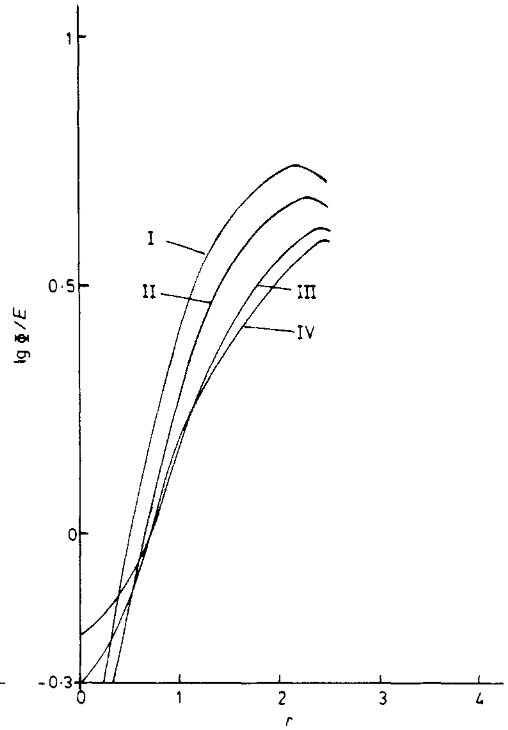

Figure 3. For counter-rotation in a 'dense' source ($m/b = 0{\cdot}40$) with different values of a/b. I—for $a/b = 0{\cdot}05$. II—for $a/b = 0{\cdot}10$. III—for $a/b = 0{\cdot}20$. IV—for $a/b = 0{\cdot}25$.

where the overhead dots indicates differentiation with respect to an affine parameter and of course $\sin\theta = 1$ for equitorial paths.

The turning points are given by the zeros of R, which from (16) are given by

$$\frac{\Phi}{E} = \frac{a(1-f) \pm (a^2 + r^2 f)^{1/2}}{f}. \tag{18}$$

The upper and the lower sign in the expression (18) holds for co-rotating and counter-rotating orbits respectively.

Equation (18) permits us to study Φ/E, the angular momentum per unit energy, as a function of the turning-point co-ordinate r, for different values of the parameters a, b, and m. For the turning point to be at the centre of the central rotating object, Φ/E shows a finite value (vanishing only for the case $a = 0$). This is unlike the classical case, where for a particle to reach the centre of symmetry, the angular momentum of the test particle should be zero. The difference apparently is due to the fact that Φ now does not vanish, for $\dot{\phi} = 0$.

The following figures show a study of Φ/E, for different values of r from $r = 0$ to $r = b$. Figure 1 is the non-rotating case ($a = 0$) for a 'dense' source ($m/b = 0{\cdot}40$), where the circular orbit is obtained at $r/b \approx 0{\cdot}72$. The Hogan metric reduces to the Schwarzschild interior solution in this case. Figure 2 shows Φ/E for various non-zero values of a for both 'dense' and 'diffuse' sources ($m/b = 0{\cdot}40$ and $0{\cdot}30$) in the case of co-rotating orbits. Figure 3 shows the nature of the curves for the particle with counter-rotation for different values of a/b in the case of 'dense' source ($m/b = 0{\cdot}40$).

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